Basics on commutative algebra and on category theory

August 2014

1 Modules

Exercise 1.1. *All vector spaces are modules. Every ring is a module over itself, and its submodules are precisely its ideals. Modules over $\mathbb{Z}$ are exactly the abelian groups up to isomorphism.*

Proof. Since all fields k are in particular rings, it follows from the defining properties that a k-module is just a k-vector space. The action of a ring A onto itself defined as just the product of elements in A makes indeed A into an A-module, and a subgroup that is closed under this action is precisely the concept of an ideal. Finally, note that given an abelian group there is a unique way of defining the action by elements of Z: $n * a$ is $a + a + ... + a$ precisely n times. □

Example 1.2. *$f : M \to N$ linear map, $ker(f)$ submodule of M and $Im(f)$ submodule of N.*

Exercise 1.3. *$f : M \to N$ linear is injective $\Leftrightarrow ker(f) = \{0\}$*

Proof. If f is injective, $x \in ker(f)$ implies $f(x) = 0 = f(0)$, so that $x = 0$. Conversely, $f(x) = f(y) \Leftrightarrow f(x) - f(y) = 0 \Leftrightarrow f(x - y) = 0 \Leftrightarrow x - y \in ker(f) = \{0\}$ so that $x = y$. □

Exercise 1.4. *M is a free module if there is a basis ($\Leftrightarrow$ there is an isomorphism $A^{(I)} \equiv M$ for some set I, $A^{(I)} = \oplus_I A \neq A^I$).*
Prove the above equivalence.

Proof. If there is a basis (x_i) with $i \in I$ then $f : M \to A^{(I)}$ given by sending each elements to its coefficients when expressed uniquely in the basis is an isomorphism. Conversely, if such isomorphism exists, then taking $x_i \in M$ such that $f(x_i) = e_i$ ($e_i(j) = \delta_{ij}$ for all $j \in I$) is easily seen to be a basis for M. □

Exercise 1.5. *If M is finitely generated and has a basis, then the basis is finite.*

Proof. Let $m_1, ..., m_N$ generate M. Now consider some basis $\{b_i\}$ where i ranges over some possibly infinite set. We can express each generator in terms of the basis, say $m_i = a_{i_1} b_{i_1} + ... + a_{i_{N_i}} b_{i_{N_i}}$ But this makes it apparent that only finitely many elements of $\{b_i\}$ are needed to generate M (namely $b_{i_{N_i}}$ where i ranges from 1 to N). □

Exercise 1.6. *If a finitely generated $\mathbb{Z}$-module has two different bases, they are the same size.*

Proof. Call the $\mathbb{Z}$-module M. Due to the previous exercise, we know every basis is finite. Choose such a basis $m_1, ..., m_N$. By a prior exercise, we know the module is isomorphic to $\mathbb{Z}^N$. Now considering the module M as a group, take the subgroup $2M = \{m \in M | \exists n \text{ such that } m = n + n\}$. Note that $M/2M$ is a group of order 2^N. But the defintion of $M/2M$ does not depend on any choice of basis, so all bases must be length N. $\square$

Exercise 1.7. *(ISOMORPHISM THEOREM) If $f : M \to N$ linear surjective map, prove that $M/ker(f)$ is isomorphic to N as modules.*

Proof. Define $\tilde{f} : M/Ker(f) \to N$ by $\tilde{f}(\tilde{m}) = f(m) \in N$. This is readily seen to be well-defined and surjective. For injectivity, $f(\tilde{m}) = 0 \Leftrightarrow f(m) = 0 \Leftrightarrow m \in ker(f)$, so that $\tilde{m} = \tilde{0}$. $\square$

1.1 Noetherian modules:

Exercise 1.8. *A noetherian, M A-module then M is noetherian if and only if M is finitely generated.*

Proof. If M is noetherian, then every submodule is finitely generated, in particular M itself. Conversely, if M is finitely generated then there exists a surjective morphism $f : A^m \to M$ for some natural m, and so by Isomorphism Theorem M is a quotient of A^m (which is Noetherian by hypothesis), and thus noetherian itself. $\square$

Exercise 1.9. *Prove the converse of Hilbert's Theorem: If $A[X]$ is noetherian $\Rightarrow$ A is noetherian.*

Proof. Applying the Isomorphism Theorem to the evaluation morphism $f : A[X] \to A$, $f(p(x)) = p(a)$ we obtain that A is isomorphic to $A[x]/ < x >$; since the latter is a quotient of a noetherian ring by hypothesis, the former is also noetherian. $\square$

1.2 Localization

Exercise 1.10. *Let $\mathfrak{p}$ be a prime ideal of A, $A_{\mathfrak{p}} = S^{-1}A$, $S = A - \mathfrak{p}$ multiplicative and δ defined as*

$$\begin{array}{rccc} \delta : & A & \to & S^{-1}A \\ & a & \mapsto & a/1 \end{array}$$

of A. $S^{-1}A$ with δ has the following universal property: Let $f : A \to B$ be a ring homomorphism then f factorizes uniquely though $S^{-1}A$ if and only if $f(S) \subset B^{\times}$ (invertible)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\delta"'] & B \\ S^{-1}A \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

$\tilde{f}$ is a ring homomorphism $\tilde{f}(a/s) = f(a)f(s)^{-1}$, $f(s) \in B^\times$.
Prove the above universal property.

Proof. We can check that $\tilde{f}$ is well-defined: if $a/s = b/t$ then there exists $u \in S$ such that $u(at-bs) = 0$ and then applying f we get $f(u) * (f(at) - f(bs)) = 0$ and thus (since $f(u)$ is a unit) $f(a)f(t) - f(b)f(s) = 0$, which then implies $f(a)f(s)^{-1} = f(b)f(t)^{-1}$, as we wanted to prove. Clearly $\tilde{f}$ makes f factor through δ. Conversely, if such a map $\tilde{f}$ exists, it has to satisfy $\tilde{f}(s/1) = f(s)$ and also since $1 = \tilde{f}(1/1) = f(s/1 * 1/s) = f(s) * \tilde{f}(1/s)$ then $f(s)$ must be in $B^\times$ and $\tilde{f}$ has to be $\tilde{f}(a/s) = f(a)f(s)^{-1}$. $\square$

Notation: B ring, $Spec(B) = \{prime\ ideals\ of\ B\}$ (Spectrum of B). If $f : A \to B$ ring homomorphism $\Rightarrow$

$$\begin{array}{llll} Spec(f): & Spec(B) & \to & Spec(A) \\ & Q & \mapsto & f^{-1}(Q) \end{array}$$

Exercise 1.11. *Show that for any ring homomorphism $f : A \to B$ that $Spec(f)$ indeed sends elements in $Spec(B)$ to elements in $Spec(A)$*

Proof. Choose some prime ideal p_B in $Spec(B)$. Now let $p_A = Spec(f)(p_B) = f^{-1}(p_B)$ Now check that p_A is an ideal. Let $a, b \in p_A$ Then $f(a), f(b)$ are in p_B and then so is $f(a) + f(b) = f(a+b)$. Then $a+b$ is in $p_A = f^{-1}(p_B)$ Now Let $a \in p_A$ and $r \in A$. Then again $f(a) \in p_B$ and $f(ra) = f(r)f(a) \in p_B$ since p_B is an ideal. Hence $ra \in p_A$ and p_A is an ideal. Now check p_A is prime. Suppose $ab \in p_A$. Then $f(ab) = f(a)f(b) \in p_B$ then either $f(a)$ or $f(b)$ by primality of p_B. Then one of a or b is in $f^{-1}(p_B)$. We also must check that $f^{-1}(p_B)$ is not the entire ring B. Supposing it is, then it contains 1_B. But then p_A would have had to contain 1_A which is a contradiction. $\square$

Proposition 1.12. *$Spec(f) : Spec(S^{-1}A) \to Spec(A)$ induces a bijection from $Spec(S^{-1}A)$ to $\{Q \in Spec(A) | Q \cap S = \emptyset\}$ where $f : A \to S^{-1}A$ is defined $f(a) = a/1$.*

Proof. Surjection: First choose some $p_A \in Spec(A)$ where $p_A \cap S = \emptyset$ Now let $p_{S^{-1}A} =$ ideal-generated-by$\{f(a)\}_{a \in p_A} = \{a/s\}_{a \in p_A, s \in S}$. Then $Spec(f)(p_{S^{-1}A}) = f^{-1}\{a/s\}_{a \in p_A, s \in S}$. Obviously $f^{-1}(p_{S^{-1}A}) \subset p_A$. Now show $f^{-1}(p_{S^{-1}A}) = p_A$. Choose $c \in f^{-1}(p_{S^{-1}A})$ Then $c/1 = a/s$ for some $a \in p_A$ and $s \in S$. Then there is a $t \in S$ such that $t(a - cs) = 0$. $0 \in p_A$ because p_A is an ideal. Then $t(a - cs) \in p_A$ but $t \notin p_A$ so by primality $a - cs \in p_A$. Then $cs = (cs - a) + a \in p_A$ because again p_A is an ideal. Then again by primality and the fact that $s \notin p_A$, $c \in p_A$. So we have $f^{-1}(p_{S^{-1}A}) = p_A$ Injection: Choose two different prime ideals of $S^{-1}A$ called $p_{S^{-1}A}$ and $q_{S^{-1}A}$. wlog there is $a/s \in p_{S^{-1}A}, a/s \notin q_{S^{-1}A}$. Then $a/1 \notin q_{S^{-1}A}$ or else $a/s \in q_{S^{-1}A} * (1/s) \subset q_{S^{-1}A}$. Similarly $a/1 \in p_{S^{-1}A} * s \subset p_{S^{-1}A}$. Then $a \in f^{-1}(p_{S^{-1}A})$ and $a \notin f^{-1}(q_{S^{-1}A})$, proving that the function $Spec(f)$ is 1-to-1. We have shown before that $Spec(f)$ sends prime ideals to prime ideals. The only thing to check is that the $Spec(f)(p)$ does not intersect S for any prime ideal p. But this is true because if $Spec(f)(p)$ contained some $s \in S$, then p would have contained $1/s$ and therefore it would have also contained 1 via $p * s \subset p$, violating primality of p. $\square$

Exercise 1.13. *Prove the above definitions are well defined and do make $S^{-1}M$ into a $S^{-1}A$-module.*

Proof. Straightforward. $\square$

Exercise 1.14. *Let S be a multiplicatively closed subset of a ring A and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = \{0\}$.*

Proof. $S^{-1}M = 0$ if and only if $m/1 = 0/1$ $\forall m \in M$ (same equivalence class) if and only if exists $s \in S$ such that $s(m * 1 - 0 * 1) = 0$ if and only if $s * m = 0$ $\forall m \in M$ i.e. $sM = \{0\}$. $\square$

1.3 Tensor product

Proposition 1.15. *The tensor exists and is unique.*

Proof. 1. uniqueness

Let $\rho : M \times N \to H'$ a another tensor product

$$\begin{tikzcd} M \times N \arrow[r, "\rho"] \arrow[d, "\delta"] & H' \arrow[dl, shift left, "\tilde{\delta}"'] \\ H \arrow[ur, shift left, "\tilde{\rho}"'] & \end{tikzcd}$$

$\tilde{\rho}$ and $\tilde{\delta}$ are linear and unique.

$$\begin{tikzcd} M \times N \arrow[r, "\rho"] \arrow[d, "\delta"] & H' \\ H \arrow[ur, "\tilde{\rho}\circ\tilde{\delta}=Id'_H{,}\tilde{\delta}\circ\tilde{\rho}=Id_H"'] & \end{tikzcd}$$

$\tilde{\delta} \circ \tilde{\rho} = Id_H \Rightarrow \tilde{\rho}$ isomorphism.

2. **Existence:**

$A^{(M\times N)}$ $(x, y) \in M \times N$,

$$e_{(x,y)} \in A^{(M\times N)} = \begin{cases} 1 & \text{in } (x,y) \text{ (coordinate)} \\ 0 & elsewhere \end{cases}$$

$\{e_{(x,y)} | (x, y) \in M \times N\}$ is a basis of $A^{(M\times N)}$.

L = submodule of $A^{(M\times N)}$ generally by the element

$$e_{(x_1,x_2,y)} - e_{(x_1,y)} - e_{(x_2,y)}$$

$$e_{(x,y_1+y_2)} - e_{(x,y_1)} - e_{(x,y_2)}$$

$x_i \in M$, $y_i \in N$.

$$e_{(ax,y)} - e_{(x,ay)},\ e_{(ax,y)} - ae_{(x,y)},\ a \in A$$

$$\begin{array}{rlll} \rho : & M \times N & \to & A^{(M\times N)}/L \\ & (x,y) & \to & e_{(x,y)}\ mod\ 2 = \bar{e}_{(x,y)} \\ & (x_1+x_2, y) & \mapsto & e_{(x_1+x_2,y)} = \bar{e}_{(x_1,y)} + \bar{e}_{(x_2,y)} = e_{(x_1,y)} + e_{(x_1,y)} + e_{(x_2,y)} \end{array}$$

f bilinear Let

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\rho"] & E \arrow[dl, shift right] \arrow[dl, shift left, "\tilde{f}"] \\ A^{(M\times N)}/L & \end{tikzcd}$$

$$\begin{tikzcd} (x,y) \arrow[r, "f"] \arrow[d] & f(x,y) \\ \bar{e}_{(x,y)} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

$\tilde{f}$ exist, $\tilde{f}'(\bar{e}_{(x,y)}) = \tilde{f}' \circ f(x,y) = f(x,y)$.
Define $\tilde{f}(\bar{e}_{(x,y)}) = f(x,y)$. Let g be the linear map defined by

$$\begin{array}{rlcl} g: & A^{(M,N)} & \to & E \\ & e_{(x,y)} & \mapsto & f(x,y) \end{array}$$

$$\sum_{(x,y)} a_{(x,y)} e_{(x,y)} \to \sum_{(x,y)} a_{(x,y)} f(x,y)$$

$L \subset ker(g)$ (Lra f bilinear).

$$e_{(x_1+x_2,y)} - e_{(x_1,y)} - e_{(x_2,y)}$$

$$\begin{tikzcd} A^{(M\times N)} \arrow[r, "g"] \arrow[d] & E \\ A^{(M,N)}/A \arrow[ur, "\tilde{\rho}"'] & \end{tikzcd}$$

$$\sum a(x,y)\bar{e}_{(x,y)} = \sum e_{(ax,y)} e_{(x,y)} = \sum (ax,y) x \otimes y$$

$\rho : M \times N \to A^{M\times N}/L$ is a tensor product of M, N over A.

$\square$

Exercise 1.16. *Show that* $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \{0\}$ *if* m,n *are coprime.*

Proof. Since m and n relatively prime there exists a linear combination $am + bn = 1$. Then $x \otimes y = (amx + bnx) \otimes y = a(mx \otimes y) + b(x \otimes ny) = a(0 \otimes y) + b(x \otimes 0) = 0 + 0 = 0$ $\square$

Proposition 1.17. *A ring, M, N A-modules*

1. $M \otimes_A A \simeq M$;
2. $M \otimes N \simeq N \otimes_A M$;
3. $(\oplus_i M_i) \otimes_A N \simeq \oplus_i (M_i \otimes_A N)$;
4. $L \otimes (M \otimes N) \simeq (L \otimes M) \otimes N$.

Proof. 1. $M \otimes_A A \simeq M$ sending $x \otimes a = (ax) \otimes 1$ to ax.
Let $M \times A \to M$ sending (x, a) to xa. It is a bilinear map such that the following diagram is commutative:

$$\begin{tikzcd} M \times A \arrow[r] \arrow[d, "\rho"'] & M \\ M \otimes_A A \arrow[ur] & \end{tikzcd}$$

2. same kind of proof: we define $M \otimes_A N \simeq N \otimes_A M$ by mapping $x \otimes y$ to $y \otimes x$.
3. We define $(\oplus_i M_i \otimes_A N \simeq \oplus_i (M_i \otimes_A N)$ by mapping $(x_i)_i \otimes y \mapsto (x_i \otimes y)_i$.

∎

Corollary 1.18. *If M is free over A with a basis $(e_\alpha)_\alpha$ then every elements g $M \otimes_A N$ can be written uniquely as $\sum_\alpha e_\alpha \otimes y_\alpha$, $y_\alpha \in N$,*

$$\begin{array}{lcl} (\oplus_i M_i) \otimes_A N & \simeq & \oplus_i (M_i \otimes_A N) \\ (x_i)_i \oplus y & \mapsto & (x_i \otimes y)_i \end{array}$$

Proof. use Proposition (c) ∎

2 Complex of modules over A

Exercise 2.1. *Let* $0 \longrightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow 0$ *be an exact sequence of A-modules. If M_0,M_2 are finitely generated, so is M_1.*

Proof. Let $m \in M_1$ and consider $f_2(m) \in M_2$. Since M_2 is finitely generated and f_2 is surjective, we can find $x_1, x_2, ..., x_r$ in M_1 such that $f_2(m) = a_1 f_2(x_1) + a_2 f_2(x_2) + ... + a_r f_2(x_r)$, so $f_2(m - a_1 x_1 - ... - a_r x_r) = 0$. By exactness, $Ker(f_2) = Im(f_1)$ so there exists a (unique since f_1 is injective) y generated by $y_1, y_2, ..., y_s \in M_0$ such that $f_1(y) = m - a_1 x_1 - ... - a_r x_r$, and then $m = a_1 x_1 + ... + a_r x_r + f_1(b_1 y_1) + ... f_1(b_s y_s)$. We see then that $x_1, x_2, ..., x_r, f_1(y_1), f_1(y_2), ..., f_1(y_s)$ generate M_1. ∎

Proposition 2.2. *Let*

$$0 \longrightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow 0$$

be a short exact sequence. Let N be a A-module then, the complex

$$0 \longrightarrow M_0 \otimes_A N \xrightarrow{f_0} M_1 \otimes_A N \xrightarrow{f_1} M_2 \otimes_A N \longrightarrow 0$$

is exact (at right), that is $M_1 \otimes_A N \twoheadrightarrow M_2 \otimes_A N$.

Proof. $\phi : M_0 \twoheadrightarrow M_1$, $M_0 \otimes N \twoheadrightarrow M_1 \otimes N$, $\sum x_i \otimes y_i \mapsto \sum_i \phi(x_i) \otimes y$. Let's show that $Im(\psi : M_0 \otimes N \to M_1 \otimes N) = ker(\phi \otimes Id_N)$ $(*)$.
We know that

$$\begin{tikzcd} M_0 \otimes N \arrow[rr, "\phi"] \arrow[d] & & M_1 \otimes N \\ (M_1 \otimes N)/Im(\phi \otimes Id) \arrow[urr, "\tilde{\phi}"'] & & \end{tikzcd}$$

$(*) \Leftrightarrow \tilde{\phi}$ is injective.

□

Corollary 2.3. *I ideal of A, M A-module then $M \otimes_A A/I \simeq M/IM$*

Proof.

$$0 \longrightarrow I \xrightarrow{f_0} A \xrightarrow{f_1} A/I \longrightarrow 0$$

$$\Leftrightarrow_{\otimes_A} \quad 0 \longrightarrow I \otimes_A M \xrightarrow{f_0} A \otimes_A M \xrightarrow{f_1} A/I \otimes_A M \longrightarrow 0$$

$I \otimes_A M \to A \otimes_A M \simeq M$ sending $\alpha \otimes x$ to $\alpha \otimes x \to \alpha x$
$\forall \alpha \in I$, $\forall x \in M$, $Im(I \otimes_A M \to M) = IM$.
Proposition implies

$$0 \longrightarrow IM \xrightarrow{f_0} M \xrightarrow{f_1} A/I \otimes_A M \longrightarrow 0$$

exact implies $M/IM \simeq A/I \otimes M$. □

Theorem 2.4. *M is flat $\Leftrightarrow$ for any injective morphism $N_1 \to N_2$ linear map of A-module then M is flat $\Leftrightarrow$ M is torsion free.*
Let M be a module on an integral domain A. M is torsion free, if $ax = 0$, $a \in A \Rightarrow a = 0$ or $x = 0$ that is equivalent to $\forall a \in A \backslash \{0\}$, $\cdot aM \to M$ sending x to ax is injective.

Proof. 1. For any integral domain A, M flat A-module implies M torsion free.
$\forall a \in A \backslash \{0\}$, $\cdot A : A \hookrightarrow A$ implies $M \simeq M \otimes_A A \to M \otimes_A A \simeq M$ injective. So that, M is torsion free.
For the converse, $\forall I$ ideal of A, $I \neq 0$, $I = aA$, $a \neq 0$, M torsion free implies $\cdot a : M \hookrightarrow M$ implies $I \otimes_A M \hookrightarrow M$.

□

Exercise 2.5. *1. Let A be a nonzero ring. Show that $A^m \simeq A^n$ then $m = n$.*

2. Could you use the same proof to show that if $f : A^m \to A^n$ is surjective, then $m \geqslant n$?

3. Could you use the same proof to show that if $f : A^m \to A^n$ is injective, then $m \leqslant n$?

Proof. Let $f : A^m \to A^n$ be an isomorphism. Then we have an exact sequence

$$0 \longrightarrow A^m \xrightarrow{f} A^n \longrightarrow 0$$

Choose any maximal ideal M of A and tensor with the field A/M

$$0 \longrightarrow (A/M)^m \xrightarrow{\tilde{f}} (A/M)^n \longrightarrow 0$$

and since A/m is a field, we have now an isomorphism of A/M vector spaces, so that we can conclude $m = n$. If we only know surjectivity, the same proof can be used by replacing the exact sequence with

$$0 \longrightarrow Ker(f) \longrightarrow A^m \xrightarrow{f} A^n \longrightarrow 0$$

, since tensor product preserves exactness from the right. If we only know injectivity, since the tensor product in general will not preserve the exact sequence, we cannot apply the same proof (even though the result is true). □

2.1 Tensor product of algebras

Proposition 2.6. *Given B, C two algebra. For any A-algebra D, and ring homomorphism $\phi : B \to D$, $\psi : C \to D$, there exists a unique ring homomorphism $B \otimes_A C \to D$.*

$$\begin{tikzcd} B \arrow[d, "i_B"] \arrow[dr, "\phi"] & \\ B \otimes_A C \arrow[r, "\theta"] & D \\ C \arrow[u, "i_C"] \arrow[ur, "\psi"] & \end{tikzcd}$$

is commutative. Here, i_B sends b to $b \otimes 1$ and c to $1 \otimes c$.

Proof.
1. $\theta(\sum_i b_i \otimes c_i) = \sum_i \phi(b_i)\psi(c_i)$.
2. θ is well defined.
3. θ satisfies the required properties. Uniqueness is clear

$$\theta(b \otimes c) = \theta((b \otimes 1) \cdot (1 \otimes c)) = \theta(b \otimes 1)\theta(1 \otimes c) = \phi(b)\psi(c)$$

$$A[X_1, ..., X_n] \otimes_A A[Y_1, ..., Y_m] \simeq A[X_1, ..., X_n, Y_1, ..., Y_m]$$

injective (free). AsA-modules, $X_1^{\alpha_1} .... X_n^{\alpha_n} \otimes Y_1^{\beta_1} ... Y_n^{\beta_m} \to X_1^{\alpha_1} ... X_n^{\alpha_n} Y_1^{\beta_1} .... Y_m^{\beta_m}$.

$$A[X_1, ..., X_n]/I \otimes_A A[Y_1, ..., Y_m]/J \simeq A[X_1, ..., X_n, Y_1, ..., Y_m]/(I, J)$$

□

2.2 Nakayama lemma

Theorem 2.7. $(a, \mathfrak{m}_0)$ *a local ring (i.e. $\mathfrak{m}$ the unique maximal ideal of A. Let M be a finitely generated A-module such that $M = \mathfrak{m}_0 M$ then $M = 0$.*

Proof. Suppose $M \neq 0$. Let $\{x_1, ..., x_n\}$ be a set of generators of M chosen n minimal. $x_1 \in M = \mathfrak{m}_0 M$. Then, there is $\alpha_1, ..., \alpha_m \in \mathfrak{m}$ such that $x_1 = \alpha_1 x_1 + ... + \alpha_n x_n$. So that, $(1 - \alpha_1) \notin \mathfrak{m} \Rightarrow 1 \cdot \alpha_1 \in A^*$

$$\Rightarrow x_1 = (1 - \alpha_1)^{-1} \alpha_2 x_2 + \cdots + (1 - \alpha_1)^{-1} \alpha_n x_n$$

$\Rightarrow \{x_1, ..., x_n\}$ generates M.
Contradiction implies that $M = 0$. □

Proposition 2.8. *Let M be a A-module then M is flat if and only if for any B prime ideal of A, $M \otimes_A B$ is flat over B if and only if for any $\mathfrak{m}$ maximal ideal of A, $M \otimes_A A_\mathfrak{m}$ is flat over $A_\mathfrak{m}$*

Proof. $\Rightarrow$ M is flat over A implies trivially that $M \otimes_A B$ flat over B. The converse is left as exercise. $\square$

Exercise 2.9. *Prove that if A is a local ring, M and N are finitely generated A-modules, and $M \otimes_A N = 0$, then one of M or N is zero.*

Proof. We have by above $M/IM \simeq A/I \otimes M$ for I the maximal ideal, so M surjects onto $k \otimes M$ (where $k = A/I$ field). Similarly for N and $k \otimes N$. Thus $M \otimes_A N = \!\!\rightarrow (k \otimes M) \otimes (k \otimes N)$ is surjective. Now the module on the right is really a k-module and the tensor product is really over k, since the tensor product of k-modules over A is really over k since factors that pull across are in equivalence classes mod m. Thus if $M \otimes_A N$ is zero, then $(k \otimes M) \otimes (k \otimes N)$ is zero. This is a tensor product of k-vector spaces of finite dimensions (since M,N are finitely generated), say m and n. Then this has dimension $mn = 0$. So $m = 0$ or $n = 0$ since $\mathbb{Z}$ is a domain! Then $M/IM = 0$ (or similary for N) implying that $M = IM$ (or likewise for N). Now the Nakayama lemma applies since M (or N) is finitely generated and A is local, to conclude $M = 0$ (or $N = 0$). $\square$

Theorem 2.10. *Let $(A, \mathfrak{m})$ be a local ring. Let M be a finitely generated A-module then M is flat if and only if M is free.*

Proof. M free $\Rightarrow$ M flat in general even if M is not finitely generated. Suppose that M is flat $M \otimes_A A/\mathfrak{m} = M \otimes_A j \simeq M/\mathfrak{m}M \to k = A/\mathfrak{m}$ (k is a field the residue field of A.) is a vector space over k of finite dimension. If $x_1, ..., x_n \in M$ are such that $\bar{x_1}, ..., \bar{x_n} \in M \otimes_A k$ is a basis.
We want to prove $\{x_1, ..., x_n\}$ is a bases of M over A.

1. If $\{x_1, ..., x_n\}$ in M such that $\{\bar{x_1}, ..., \bar{x_n}\}$ genrates $M \otimes k$ implies $\{x_1, ..., x_n\}$ generates M.
2. If $\{x_1, ..., x_n\}$ in M such that $\{\bar{x_1}, ..., \bar{x_n}\}$ is free implies $\{x_1, ..., x_n\}$ is free.

1. Suppose that $\{\bar{x_1}, ..., \bar{x_n}\}$ generates $M \otimes k$. Let $N = M/(x_1A + ... + x_nA)$, $N/\mathfrak{m}N \simeq N \otimes_A k = M \otimes k/(\bar{x_1}k + ... + \bar{x_n}k) = 0$ implies $N = \mathfrak{m}N$, since N is finitely generated Nakayama lemma implies that $N = 0$, so that $M = x_1A + \cdots + x_nA$.
2. One can suppose n is the smallest integer such that there is $\{x_1, ..., x_n\}$ in M not free, with $\{\bar{x_1}, ..., \bar{x_n}\}$ free in $M \otimes k$. There is $a_1, ..., a_n \in A$ such that $\sum_{i=1}^n a_i x_i = 0$ at least one $a_i \neq 0$.

$$0 \longrightarrow ker(f) \longrightarrow A^n \xrightarrow{f} A$$

$(b_1, ..., b_n)$ implies that $\sum_{i=1}^n b_i a_i$.

M flat over A implies $0 \longrightarrow ker(f) \otimes_A M \longrightarrow M^n \xrightarrow{f_M} M$.
$(x_1, ..., x_n) \in ker(f_M) = Im(ker(f) \otimes_A M, M^n)$ $(y_1, ..., y_n) \mapsto \sum_{i=1}^n a_i y_i$.
$b_j \in Ker(f) \subset A^n$ implies $(x_1, ..., x_n) \in M^n$. $b_j = (b_{1j}, ..., b_{nj})$, $\sum_j b_j \otimes y_j = \sum_j (b_{1j} \otimes y_j, ..., b_{nj} \otimes y_j)$.
$(x_1, ..., x_n) = (\sum_j b_{1j} y_j, \sum_j b_{2j} y_j, ...)$.
$x_1 = \sum_j b_{1j} y_j$ in $M \otimes k$ implies $\bar{x_1} = \sum_j \bar{b_{1j}} \bar{y}_j$, $b_j \in ker(f)$. So that, at least one $\bar{b_{1j}} \neq 0$.

Suppose $\bar{b_{11}} \neq 0 \Rightarrow b_{11} \notin \mathfrak{m}$. (There is $a_1, ..., a_n \in A$, such that $\sum_{i=1}^{n} a_i x_i = 0$ at least one $a_i \neq 0$).

$$b_{11}a_1 + b_{21}a_2 + ... + b_{n1}a_n = 0$$
$$a_1 + c_2a_2 + ... + c_na_n = 0, \; c_i = \overrightarrow{b_{n1}b_{i1}}$$

If $n = 1$ then $a_1 = 0$.
If $n \leqslant 1$ the $a_2(x_2 - c_1x_1) + \cdots + a_n(x_n - c_nx_1) = 0$, $\{\bar{x_2} - \bar{c_1}\bar{x_1}, \ldots, \bar{x_n} - \bar{c_n}\bar{x_1}\}$ is free $\Rightarrow a_i = 0$, then $\{a_1, .., a_n\}$ free in M tensor, M is free.

□

Exercise 2.11. *If M and N are flat A-modules, then so is $M \otimes_A N$.*

Proof. If $0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$ exact sequence of A-modules, then since N is exact:

$$0 \longrightarrow N \otimes M_0 \longrightarrow N \otimes M_1 \longrightarrow N \otimes M_2 \longrightarrow 0$$

. Now, since M is exact:

$$0 \longrightarrow M \otimes (N \otimes M_0) \longrightarrow M \otimes (N \otimes M_1) \longrightarrow M \otimes (N \otimes M_2) \longrightarrow 0$$

But since $M \otimes (N \otimes P) \simeq (M \otimes N) \otimes P$ we conclude

$$0 \longrightarrow (M \otimes N) \otimes M_0 \longrightarrow (M \otimes N) \otimes M_1 \longrightarrow (M \otimes N) \otimes M_2 \longrightarrow 0$$

so $M \otimes_A N$ is flat. □

3 Hilbert Nullstellensatz

Exercise 3.1. *Explain how to deduce the Weak Nullstellensatz from the Strong Nullstellensatz.*

Proof. If I is taken to be maximal in the Strong Nullstellensatz, then the intersection contains only one element: I itself. □

4 Categories

Definition 4.1. *A category $\mathcal{C}$ consists of:*

1. *A " collections" of objects: $ob(\mathcal{C})$;*
2. *$\forall X, Y \in Ob(\mathcal{C})$ a "collections" of objects: of sophisms : $Mor(X, Y)$ (morphism from X to Y), such that $\forall X, Y, Z \in ob(\mathcal{C})$*

$$\begin{array}{lcl} Mor(X,Y) \times Mor(Y,Z) & \rightarrow & Mor(X,Z) \\ (f,g) & \mapsto & g \circ f \end{array}$$

3. *+ associativity.*

4. $\forall x \in ob(\phi)$, $\exists Id_X \in Mor(X,X)$ *such that* $f : X \to Y$, $f \circ Id_X = f$, $Id_Y \circ f = f$.

Example 4.2. 1. *sets:*
— *objects: sets*
— $Mor(X,Y)$*: maps from* X *to* Y.

2. *groups*
— *objects: groups;*
— $Mor(X,Y) = \{$ *group homomorphism from* X *to* $Y\}$

3. *A ring, A-Mod*

Definition 4.3. 1. *A category* $\mathcal{C}$ *with* $ob(\phi) =$ *one element is called a monoid.*

2. *A category is locally small if* $\forall X, Y \in ob(\mathcal{C})$, $Mor(Y,X)$ *is a set category is small, if* $ob(\mathcal{C})$, $Mor(X,Y)$ *are sets.*

3. $ob(\mathcal{C}) = \{X\}$, $Mor(X,X) = Id_X$ *trivial monoid*

4. *Opposite category=* $\mathcal{C}^{op}$ *of a given category* $ob(\mathcal{C}^{op}) = ob(\mathcal{C})$ *and* $Mor_{\mathcal{C}^{op}}(X,Y) = Mor_{\mathcal{C}}(Y,X)$.

Definition 4.4. 1. $f : X \to Y$ *is an isomorphism if* $\exists g : Y \to X$ *such that* $f \circ g = Id_Y$ *and* $g \circ f = Id_X$.

2. $f : X \to Y$ *is a monomorphism if* $\forall Z$, $\forall$ $Z \overset{g}{\underset{h}{\rightrightarrows}} Z$ *such that* $f \circ g = f \circ h \Rightarrow g = h$.

3. $f : X \to Y$ *is an epimorphism if* $\forall Z \in ob(\mathcal{C})$, $\forall$ $X \xrightarrow{f} Y \overset{g}{\underset{h}{\rightrightarrows}} Z$ *such that* $g \circ f = h \circ f \Rightarrow g = h$.

Example 4.5. 1. *In sets, groupes A-Mod, monomorphism are injection and epimorphism are surjection.*

2. *In topology, monomorphism + epimorphism does not imply isomorphism in general.*

4.1 Functors

Definition 4.6. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *consist of*

1. $F : ob(\mathcal{C}) \to ob(\mathcal{D})$,
2. $X, Y \in Ob(\mathcal{C})$, $F : Mor(X,Y) \to Mor(F(X), F(Y))$ *such that* $F(Id_X) = Id_{F(X)}$
3. $X \xrightarrow{f} Y \xrightarrow{g} Z$
4. $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$;
5. $F(g \circ f) = F(g) \circ F(f)$

Example 4.7. 1. *Forgetful functor*

$$\begin{array}{lcl} Groups & \to & Sets \\ X & \mapsto & X \\ Mor(X,Y) & \mapsto & Mor(X,Y) \end{array}$$

2. $A \to B$ *ring homomorphism*

$$\begin{array}{lll} A-Mod & \to & B-Mod \\ M & \mapsto & M \otimes_A B \\ Mor(M,N) & \mapsto & Mor(M \otimes_A B, Y \otimes_A B) \\ u & \mapsto & u \otimes Id_B \end{array}$$

Definition 4.8. *A contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ *is a covariant functor* $\mathcal{C} \to \mathcal{D}^{op}$.

Example 4.9. $\mathcal{C}$ *small category,* $X \in ob(\mathcal{C})$,

$$\begin{array}{lll} h_X : \mathcal{C} & \to & Sets \\ Y & \mapsto & Mor(Y, X) \end{array}$$

;

—

$$\begin{array}{lll} Mor(Y,Z) & \to & Mor(Mor(Z,X), Mor(Y,X)) \\ f : Y \to Z & \mapsto & g \mapsto g \circ f \end{array}$$

— $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{G}$, $G \circ F$ *composite associative*
— *Category of the Category:* $\underline{Cat}$ *object= categories and morphism= functor*

Definition 4.10. $F : \mathcal{C} \to \mathcal{D}$ *functor*

1. F *is full if* $\forall X, Y \in ob(\mathcal{C})$, $Mor(X,Y) \twoheadrightarrow Mor(F(X), F(Y))$;
2. F *is faithful if* $Mor(X,Y) \hookrightarrow Mor(F(X), F(Y))$;
3. F *is fully faithful if it is full and faithful.*
4. *A subcategory* $\mathcal{C}$ *of* $\mathcal{D}$, $ob(\mathcal{C}) \subset ob(\mathcal{D})$,

   $X, Y \in Ob(\mathcal{C}), Mor_{\mathcal{C}}(X,Y) \subset Mor_{\mathcal{D}}(X,Y), Mor_{\mathcal{C}}(X,Y) \times Mor_{\mathcal{C}}(Y,Z) \to Mor_{\mathcal{C}}(X,Z)$
5. *full subcategory* $\mathcal{C}$ *of* $\mathcal{D}$, $Mor_{\mathcal{C}}(X,Y) = Mor_{\mathcal{C}}(X,Y)$.

Example 4.11. — *group subcategory of* $Sets$;
— Mod_A *subcategory of* $Groups$;
— $Mod_{\mathbb{Q}}$ *full subcategory of* $Groups$;
— $E, F\mathbb{Q}$*-vector space,* $f : E \to F$ *homomorphism of groups*
$\mathbb{Z}$*-linear* $\Rightarrow f(x) = f(n/nx) = nf(1/nx)$, *so that* $1/nf(x) = f(1/nx)$.

4.2 Morphism of functor

Definition 4.12. 1.

$$\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$$

functors, a morphism from F *to* G *(a natural transformation given by* $\forall X, Y \in Ob(\mathcal{C}), \alpha_X : F(X) \to G(X)$, $\alpha_X \in Mor(F(X), G(X))$ *such that* $\forall \phi : X \to Y$,

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"'] & G(Y) \end{tikzcd}$$

2. α *is an isomorphism if* α_X *is an isomorphism* $\forall X$.

3. $F : \mathcal{C} \to \mathcal{D}$ *is an equivalence of categories if* $\exists G : \mathcal{D} \to \mathcal{C}$ *such that* $G \circ F : \mathcal{C} \to \mathcal{C}$ *is "naturally isomorphic" to* Id_ϕ.

Example 4.13. 1. V_K *category of vector spaces over* K*, finite dimensional.*

$$V_K \to V_K$$

$ob(V_K) = \{(E, f) | E \text{ finite dimensional vector space over } K,\ Aut_K(E)\}$
$(E, f) \to (F, g)$

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "f"'] & F \arrow[d, "g"] \\ E \arrow[r, "h"] & E \end{tikzcd}$$

$$\begin{array}{llcl} F: & V_K & \to & V_K \\ & (E, f) & \mapsto & (E^*, (f^*)^{-1}) \end{array}$$

2. F *is not isomorphic to* Id_{V_K} *(among not isomorphic to this)*

$$\begin{array}{llcl} G: & V_K & \to & V_K \\ & (E, f) & \mapsto & (E^{**}, (f^*)^*) \end{array}$$

G *isomorphic to* Id_{V_K}.

Definition 4.14. $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *G is adjoint to F if* $\forall X \in Ob(\mathcal{C}), \forall \in Ob(\mathcal{D}$,

$$\alpha_{X,Y} : Mor_{\mathcal{D}}(F(X), Y) \simeq Mor_{\mathcal{D}}(X, G(Y))$$

Example 4.15. $A \to B$ *ring hom,*

$$\underline{Mod}_A \xrightarrow{F} \underline{Mod}_B \xrightarrow{G} \underline{Mod}_A$$

$$M \longrightarrow M \otimes_A B$$

$$N \longrightarrow N$$

$$Hom_B(M \otimes_A B, N) \simeq Hom_A(M, N)$$

F *is adjoint to* G.

5 Presheaf

Definition 5.1. *X topological space. A pre sheaf $\mathcal{F}$ on X of groups is functor (contravariant) from $\underline{X}$ to Groups, where $\underline{X}$, $ob(\underline{X})$ = open subset of X.*
U, V open subsets of X

$$Mor(U, V) = \begin{cases} \text{canonical inclusion of } U \text{ in } V \text{ if } U \subset V \\ \emptyset \text{ otherwise} \end{cases}$$

$\forall U \subseteq X$ *open subset*

1. $U \mapsto \mathcal{F}(U)$ *group*
2. *if* $U \subseteq V \Rightarrow \exists \rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ *homomorphism of groups such that:*
   (a) $f_{UU} = Id_{\mathcal{F}(U)}, \forall U$;
   (b) $U \subseteq V \subseteq W$ *open subset*

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "\rho_{WV}"] \arrow[d, "\rho_{WV}"'] & \mathcal{F}(U) \\ \mathcal{F}(V) \arrow[ur, "\rho_{VU}"'] & \end{tikzcd}$$

   (c) $\mathcal{F}(\emptyset) = \{0\}$.

Elements of $\mathcal{F}(V)$ *are called a solution of* $\mathcal{F}$ *on* V;
If $U \subseteq V$, $s \in \mathcal{F}(V)$, $s|_U := \rho_{VU}(s)$ *is called the restriction of* s *to* U.

Example 5.2. X *topological space* $\forall U \subseteq X$, $\mathcal{F}(U) = C(U, \mathbb{R})$. *If* $U \subseteq V$, $\rho_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$ *restriction maps;*

Constant pre sheaf: we fixe a group G; $\mathcal{F} : U \to G$, $\emptyset \mapsto \{0\}$ *(if* $U \neq \emptyset$*)*, $\rho_{UV} = Id_G$ *if* $V \neq \emptyset$ *and* $\rho_{U\emptyset} = 0$ *maps is a presheaf.*

5.1 Sheaves

Definition 5.3. *A sheaf on X is a pre sheaf* $\mathcal{F}$ *on X such that:*

1. *(Uniqueness condition)* $\forall U \subseteq X$, $\forall \{U_i\}_i$ *open covering of* U,
   $\begin{array}{ccc} \mathcal{F} & \to & \coprod_i \mathcal{F}(U_i) \\ s & \mapsto & (s|_{U_i})_i \end{array}$ *is injective.*
2. *(glueing condition):* U, $\{U_i\}_i$ *then* $\forall (s_i)_i$, $s_i \in \mathcal{F}(U_i)$ *such that* $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.
   There exists (unique) in $\mathcal{F}(U)$ *such that* $s|_{U_i} = s_i$, $\forall i$.

Remarque 5.4. 1. *Condition1. and 2.* $\Leftrightarrow$ $0 \longrightarrow \mathcal{F}(U) \xrightarrow{r} \coprod_i \mathcal{F}(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \coprod_{i,j} \mathcal{F}(U_i \cap U_j)$

$$s \longrightarrow (s|_{U_i})_i$$

$$\begin{cases} p((s_i)_i) &= (s_{i(U_i \cap U_j)})_{(i,j)} \\ q((s_i)_i) &= (s_j)_{U_i \cap U_j} \end{cases}$$

2. *exact* $\Leftrightarrow \begin{cases} r \text{ injective} \\ p(\alpha) = q(\alpha) \Rightarrow \alpha \in Im(r), \alpha \in \coprod_i \mathcal{F}(U_i) \end{cases}$.

$\mathcal{F}$ presheaf of group, therefore one can replace (p, q) by $(s_i)_i \mapsto (s_i)|_{U_i \cap U_j} \cdot (s_j)_{(U_i \cap U_j)^{-1}}{}_{(i,j)}$.

Example 5.5. 1. $U \mapsto C(U, \mathbb{R})$ *is a sheaf;*

2. *constant pre sheaf is not a sheaf because if* $U \subseteq X$ *not connected* $U = U_1 \coprod U_2$, $G = G(U_1 \coprod U_2)$ *so that* $G = G(U_1)$ *and* $G = G(U_2)$. $\forall s \in G(U_1), t \in G(U_2)$, $s \neq t \Rightarrow s|_{U_1 \cap U_2} = t|_{U_1 \cap U_2}$ *because* $U_1 \cap U_2 = \varnothing$. *But there is a section* $h \in G(U)$ *such that* $h|_{U_1} = s$ *and* $h|_{U_2} = t$.
   $X = \mathbb{C}$, $U \to \{f \in C^0(U, \mathbb{C}) | f(z) = \sum_{n \geqslant 0} a_n z^n$ *absolutely convergent serie*$\}$ *presheaf not a sheaf.*

5.2 Morphisms of pre sheaves

$\mathcal{F}$, $\mathcal{G}$ presheaves in X a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is given by: $\forall U set X$, $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ homomorphism of groups such that $\forall V \subseteq U$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\rho_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\phi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

ϕ is natural transformation of functors. A morphism of sheaves is a morphisms of pre sheaves between sheaves.
We denote by $\underline{Psh}_X$ the category of pre sheaves on X and $\underline{Sh}_X$ the category of sheaves on X, it is a full subcategory $\underline{Psh}_X$.
$\mathcal{C}^\infty(\mathbb{R})$ sheaf of $\mathcal{C}^\infty$ functions on X.

$$\begin{array}{ccc} \mathcal{C}^\infty_X(\mathbb{R}) & \to & \mathcal{C}^\infty_X(\mathbb{R}) \\ f & \mapsto & f' \end{array}$$

derivation, this is an homomorphism.

5.3 Subsection of a sheaf $\mathcal{F}$

A sub sheaf is a sheaf $\mathcal{G}$ such that $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ subgroup $\forall \phi$.

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r, phantom, "\subset"] \arrow[d, "\rho_{UV}"'] & \mathcal{F}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{G}(V) \arrow[r, phantom, "\subset"] & \mathcal{F}(V) \end{tikzcd}$$

5.4 Sheaf of ideals

- θ a sheaf of commutative unitary ring, sheaf of θ-modules
- $\forall U$
  - $\mathcal{F}(U)$ is a $\theta(U)$-module
  - $V \subseteq U$, $a \in \theta(V)$, $s \in \mathcal{F}(U) \Rightarrow (as)|_V = a|_V s|_V$
  - $\mathcal{F}$ is a sheaf.
- sheaf of ideals ρ of θ is sub sheaf of θ as sheaf of θ modules (i.e. $\rho(U)$ ideal of $\theta(U)$, $\forall U$).

Example 5.6. *1. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of groups then*

$$ker(\phi) : \begin{array}{lcl} U & \to & ker(\phi(U)) \\ \mathcal{F}(U) & \mapsto & \mathcal{G}(V) \end{array} \Rightarrow ker(\phi) \text{ is a sheaf}$$

What would be $Im(\phi)$, $\phi : \mathcal{F} \to \mathcal{G}$?
Natural way :

$$U \to Im(\phi(U)) : \mathcal{F}(U) \to \mathcal{G}(U)$$

This defines a sub pre sheaf. But in general it is not a sheaf. When $\mathcal{F}$ and $\mathcal{G}$ are sheaves.

2. $\mathcal{H}$ = holomorphic function on $\mathbb{C}$. $U \subseteq \mathbb{C}$, $\mathcal{H}(U) = \{holomorphism\ functions\ U \to \mathbb{C}\}$, $\mathcal{G}(U) = \{holomorphic\ function : U \to \mathbb{C}^*\}$, $exp : \mathcal{H}(U) \to \mathcal{G}(U)$, $exp : \mathcal{H} \to \mathcal{G}$.
$Im(exp)$ is not a sheaf, $g_i \in \mathcal{H}(U_i), f_i = exp(g_i)$, $\mathcal{U} = \cup_i \mathcal{U}_i$, $f_i \in \mathcal{G}(U_i)$, $f_{i_{U_i \cap U_j}} = f_j|_{U_i \cap U_j}$ does not implies $\exists g \in \mathcal{H}(U)$ such that $exp(g)|_{U_i} = f_i$.

5.5 Inductive limite

Definition 5.7. *We are given a set I of indexes. $\forall i \in I$, a group E_i such that*

— *I is partially ordered and $\forall i, j \in I$, $\exists k \in I$, $i \leqslant k$, $j \leqslant k$;*

— *$\forall i, j \in I$, $i \geqslant j$, $\phi_{i,j} : E_i \to E_j$ group homomorphism. If $i \leqslant j \leqslant k$;*

$$\begin{tikzcd} E_i \arrow[r, "\phi_{ij}"] \arrow[d, "\phi_{ik}"'] & E_j \arrow[d, "\phi_{jk}"] \\ E_k & {} \end{tikzcd}$$

— *$\phi_{ii} = Id_{ii}$.*

$\varinjlim E_i$ inductive limit is a group. The homomorphism $E_k \to \varinjlim_i E_i$ $\forall k$ such that if $k \leqslant j$,

$$\begin{tikzcd} E_k \arrow[r, "\phi"] \arrow[d] & E_j \arrow[dl, "\phi_j"] \\ \varinjlim_i E_i & \end{tikzcd}$$

(Universal property)
If G is a group and $\forall i$, $\psi_i : E_i \to G$ group homomorphism such that $\forall i \leqslant j$,

$$\begin{tikzcd} E_i \arrow[r, "\phi_{ij}"] \arrow[d, "\phi_i"'] & E_j \arrow[dl, "\psi_j"] \\ G & \end{tikzcd}$$

then there exists a unique factorization $\psi : \varinjlim_i E_i \to G$, $\forall i$,

$$\begin{tikzcd} E_i \arrow[r, "\psi_i"] \arrow[d, "\phi_i"'] & G \arrow[dl, "\psi"] \\ \varinjlim_i E_i & \end{tikzcd}$$

Proposition 5.8. $\varinjlim_i E_i$ *exists and is unique up to unique isomorphism.*

Proof. — Uniqueness: standard argument (up)
existence: Set $\coprod_{i\in I} E_i$, $\sim x, y \in \coprod_i E_i \Rightarrow x \in E_{i_0}$ *and* $u \in E_{j_0}$.
$x \sim y$ if $\exists k_0 \geqslant i_0$ and j_0 such that $\phi_{i_0k_0}(x) = \phi_{j_0k_0}(g)$. Take $k_2 \geqslant k_0, k_1$,

$$\begin{tikzcd} E_{i_0} \arrow[dr] & & E_{j_0} \arrow[dl] \arrow[dr] & & E_{k_0} \arrow[dl] \\ & E_{k_0} \arrow[dr] & & E_{k_1} \arrow[dl] & \\ & & E_{k_2} & & \end{tikzcd}$$

$\coprod_i E_i/\sim$, one can check $\phi_k : E_k \to \coprod_i E_i \to \coprod E_i/\sim$, $\{\phi_k : E_k \to \coprod_i E_i/\sim\}_k$ is an inductive limit. $\square$

Example 5.9. *X topological space, $\mathcal{F}$ pre sheaf on X, $x_0 \in X$, $I = \{open\ subsets\ of\ X\ containing\ x_0\}$, $U, V \in I$, $U \leqslant V$ if $V \subseteq U$, $E_U = \mathcal{F}(U)$, $U \leqslant V \Rightarrow V \subseteq U$,*

$$\begin{tikzcd} \phi_U|_V : E_U \arrow[r] \arrow[d, equal] & E_V \arrow[d, equal] \\ \mathcal{F}(U) \arrow[r, "\rho_{UV}"] & \mathcal{F}(V) \end{tikzcd}$$

$U, V \in I \Rightarrow U \leqslant U \cap U$ and $V \leqslant U \cap V$.

Definition 5.10. $\mathcal{F}_{x_0} := \varinjlim x_0 \in U\mathcal{F}(\mathcal{U})$*: the stalk of $\mathcal{F}$ at x_0.*

Example 5.11. $\mathcal{F} = C^0_X(\mathbb{R})$, $x_0 \in X$, $\mathcal{F}_{x_0} = \{[f] | f \in C^0(U, \mathbb{R}), x_0 \in U\}$
— $[f] = [g]$ *if* $g = f$ *on some open neighborhood of* x_0*;*
— $s \in \mathcal{F}(U)$, $x_0 \in U$, *denote by* s_{x_0} *the image of* s *in* $\mathcal{F}_{x_0}$.

Lemma 5.12. *Let $\mathcal{F}$ be a sheaf on X then $\forall U \subseteq X$, the map*

$$\begin{array}{lll} \mathcal{F}(U) & \to & \coprod_{x\in U} \mathcal{F}_x \\ s & \mapsto & (s_x)_{x\in U} \end{array}$$

is injective

Proof. Let $s, t \in \mathcal{F}(U)$ such that $s_x = t_x$, $\forall x \in U$. For any $x \in U$, $s_x = t_x$ mean, $\exists x \in V_x$ such that $s|_{V_x} = t|_{V_x}$. $V = \cup_{x\in U} V_x \Rightarrow s = t$. $\square$

Proposition 5.13. *Let $\mathcal{G} : \mathcal{F} \to \mathcal{G}$ morphism of sheaves. For any $x \in X$, suppose that $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism then ϕ is an isomorphism.*

Proof. Definition of ϕ_x
$\phi : \mathcal{F} \to \mathcal{G}$ morphism of presheaves, $x_0 \in X$, $\phi_{x_0} : \mathcal{F}_{x_0} \to \mathcal{G}_{x_0}$, $\forall x_0 \in U$, $x_0 \in V \subseteq U$

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[d, "can"] \arrow[dr, "\rho_{UV}"] & \mathcal{G}(U) \arrow[l, "\phi(U)"'] \arrow[r, "can"] \arrow[dr, "can"] & \mathcal{G}_{x_0} \\
\mathcal{F}_{x_0} & \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \arrow[u]
\end{tikzcd}$$

$s \in \mathcal{F}_{x_0} \to \mathcal{G}_{x_0} \Rightarrow \exists x_0 \in U,\ t \in \mathcal{F}(U)$, such that $s = t_{x_0}$.

$$\varinjlim limE_i = \coprod E_i/sim$$

$t \in \mathcal{F}(U)$, $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ $(\phi(U)(t))_{x_0}$ such that $s = t_{x_0}$.
$U \subseteq X$, $\phi(U) : \mathcal{F}(U) \simeq \mathcal{G}(U)$

1. $\phi(U)$ injective

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\prod_{x \in C} \mathcal{F}_x \arrow[r] & \prod_{x \in C} \mathcal{G}_x
\end{tikzcd}$$

   $t \in \mathcal{G}(U)$, $\forall x \in U$, $t_x = \phi_x(s(x))$, $s(x) \in \mathcal{F}_x$, $\exists s_x \in \mathcal{F}(U_x)$ depending on x such that $sx = s(x)$.
   $t_x = \phi_x(s)$, $t_x = \phi(V_x)(s)_x = \phi_x(s_x)$ in $\mathcal{G}(V_x)$. t and $\phi(V_x)(s)$ some stalk at x then $\exists x \in W_x \subseteq V_x$ on which $t|_{W_x} = s|_{W_x}$. $U = \cup_{x \in U} W_x$ Be careful $s = s^x$ depend on x, $(s^x|W_x)_x$, $s^x|_{W_x \cap W_y} s^x|_{W_x \cap W_y}$ glueing condition $\Rightarrow \exists s \in \mathcal{F}(U)$, $s|W_x = s^x$, $\forall x$,
   $\phi(U)(s)_x = t_x$, $\forall x \in C \Rightarrow \phi(U)(s) = t \Rightarrow \phi(U)$ surjective.

□

5.6 Sheaf associated to a presheaf

Let $\mathcal{F}$ be a presheaf on X of opposite sets.

Proposition 5.14. *There is a unique pair $\mathcal{F}^+$ sheaf on X and $\theta : \mathcal{F} \to \mathcal{F}^+$ a morphism of presheave such that $\forall \mathcal{F} \to \mathcal{G}$ from $\mathcal{F}$ to a sheaf $\mathcal{G}$ has an unique factorization*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\psi"] \arrow[d, "\theta"'] & \mathcal{G} \\
\mathcal{F}^+ \arrow[ur, "\psi^+"'] &
\end{tikzcd}$$

$$\forall x \in X,\ \mathcal{F}_x \simeq \mathcal{F}^+_x$$

Definition 5.15. *1. If $f : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves then $Im(\phi)$ is the sheaf associated to the pre sheaf $U \mapsto Im(\phi(U))$.*

2. *If* $\mathcal{F} \subset \mathcal{G}$ *sub sheaf then* $\mathcal{G}/\mathcal{F}$ = *sheaf associated to the pre sheaf* $U \to \mathcal{G}(U)/\mathcal{F}(U)$;
3. *We say that a morphism* $\phi : \mathcal{F} \to \mathcal{G}$ *is surjective if* $Im(\phi) = \mathcal{G}$ ($\Leftrightarrow \phi_x : \mathcal{F}_x \twoheadrightarrow \mathcal{G}_x$ $\forall x \in X$) *in* $\underline{sh}_X$, ϕ *is surjective if and only if* ϕ *is an epimorphism.*
4. *(Exact sequence)* $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ *mean* $ker(\psi) = Im(\phi)$

$$\Leftrightarrow \forall x \in X,\ \ \mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x \ \ exact;$$

5. $f : X \to Y$ *continuou,* $\mathcal{F}$ *sheaf on* X, $\mathcal{G}$ *sheaf on* $Y \Rightarrow f_*\mathcal{F}$ *direct image of* $\mathcal{F}$ *by* f.
6. $f_*\mathcal{F} : U \subseteq Y \to \mathcal{F}(\phi^{-1}(V))$, $U \subseteq X$ *sheaf associated* $U \to \varinjlim_{f(U)\subset W, W \text{ open in } Y} \mathcal{G}(W)$. *Presheaf but in general not a sheave*

$$\forall x \in X,\ (f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$$